Math 279 Lecture 28 Notes

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1 Hopf Algebras for Constructing Regularity Structures

1.1 Building up to Hopf algebras

Here is the algebraic part of the story: We learn how to build an important group of transformations $\{\Gamma_g : g \in G_0\} = G$ for a Hopf algebra. It is this group that yields our group G in our regularity structure. Here are the first few steps:

Definition 1.1.

1. **Algebra**: Given a field k and a k-vector space A, by a **product**, we mean a linear map $m : A \otimes A \to A$ that is associative. We also have a unit $\mathbf{1} \in A$, which we also write as $\mathbf{1} : k \to A$ by $\mathbf{1}(\lambda) = \lambda \mathbf{1}$.

2. **Coalgebra**: With A as above, we now have a **coproduct**, a linear $\Delta : A \to A \otimes A$ (we can think of this as $\Delta a = \sum_i a^i \otimes \widehat{a}^i$ with $a^u, \widehat{a}^i \in A$). This is **coassociative**, which means
$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta : A \to A \otimes A \otimes A.$$
We also have a **counit** $\mathbf{1}' : A \to k$ such that
$$\Delta a = \sum_i a^i \otimes \widehat{a}^i \implies \sum_i \mathbf{1}'(a^i)\widehat{a}^i = \sum_i \mathbf{1}'(\widehat{a}^i)a^i = a.$$
(In fact, Δ being a coproduct is equivalent to $\Delta^* : A^* \otimes A^* \to A^*$ is a product.)

3. **Bialgebra**: This is $(A; m, \mathbf{1}; \Delta, \mathbf{1}')$ with $(A; m, \mathbf{1})$ an algebra, $(A, \Delta, \mathbf{1}')$ a coalgebra, and compatibility between these two structures: First, define a product $m_2 : A \otimes A \otimes A \otimes A \to A \otimes A$ by extending the following bilinear map:
$$m_2(a \otimes b, a' \otimes b') = m(a, a') \otimes m(b, b').$$
The compatibility is that $\Delta : (A, m) \to (A \otimes A, m_2)$ is a morphism with respect to the algebra structures:
$$\Delta(m(a, b)) = m_2(\Delta(a), \Delta(b)).$$
We may also write this as $\Delta(a \cdot b) = (\Delta a) \cdot_2 (\Delta b)$.

4. **Convolution product**: If $(A; m, \mathbf{1})$ is an algebra, $(C; \Delta, \mathbf{1}')$ is a coalgebra, let $\mathcal{L}_k(C, A)$ be the set of linear maps $C \to A$ (for example $\mathcal{L}(C, k) = C^*$). Then we can turn $\mathcal{L}_k(C, A)$ into an algebra by

$$(f \star g)(c) = (m \circ (f \otimes g) \circ \Delta)(c).$$

Indeed,

$$\Delta c = \sum_i c^i \otimes \widehat{c}^i \implies (f \star g)(c) = \sum_i f(c^i) \cdot_m g(\widehat{c}^i),$$

where $a \cdot_m b = m(a, b)$. Here is our unit element: $\mathbf{1}_A \circ \mathbf{1}'_C : C \to A$. This is nothing other than $(\mathbf{1}_A \circ \mathbf{1}'_C)(c) = \mathbf{1}_C(c)\mathbf{1}_A$.

Remark 1.1. If $A = k$, then $\mathcal{L}(C, k) = C^*$, and $f \star g = f \star_\Delta g = \Delta^*(f, g)$. In particular,

$$\Delta c = \sum_i c^i \otimes \widehat{c}^i \implies (f \star g)(c) = \sum_i f(c^i) g(\widehat{c}^i).$$

Example 1.1. Let $(G, \cdot, 1)$ be a group, and let k be a field. Then $A = kG = \operatorname{span}_k\{g : g \in G\}$ have multiplication $m(g_1, g_2) = g_1 \cdot g_2$, unit $\mathbf{1} = 1$, coproduct $\Delta(g) = g \otimes g$, and counit $\mathbf{1}'(\sum_i \lambda_i g_i) = \sum_i \lambda_i$. Then $(kG; m, 1; \Delta, \mathbf{1}')$ is a bialgebra.

Definition 1.2.

5. **Hopf algebra**: By a **Hopf algebra**, we mean a bialgebra $(H; m, \mathbf{1}; \Delta, \mathbf{1}')$ for which we can find a linear $S : H \to H$ such that if $\star$ is the convolution product for $\mathcal{L}_k(H, H)$, then

$$(S \star \mathrm{id}_H)(h) = (\mathrm{id}_H \star S)(h) = \mathbf{1}'(h)1,$$

where $\mathrm{id}_H, S : H \to H$. Equivalently,

$$\Delta h = \sum_i h^i \otimes \widehat{h}^i \implies \sum_i h^i \cdot_m S(\widehat{h}^i) = \sum_i S(h^i) \cdot_m \widehat{h}^i = \mathbf{1}'(h)\mathbf{1}.$$

Example 1.2. Continuing our previous example, our kG is a Hopf algebra with $S(g) = g^{-1}$ for $g \in G$.

1.2 Constructing a group of transformations from a Hopf algebra

Here is the next step:

6. Let $(H; m, \mathbf{1}; \Delta, \mathbf{1}'; S)$ be a Hopf algebra, and assume that we have a pairing of H^* and H (not necessarily the dual space pairing). Then $(H^*; \Delta*, (\mathbf{1}')^*; m^*, \mathbf{1}^*; S^*)$ is again a Hopf algebra. Given $g \in H^*$, set $\Lambda_g : H^* \to H^*$ by $\Lambda_g(f) = f \cdot_{\Delta^*} g$. We write $\Gamma_g : H \to H$ for Λ_g^*:

$$\langle \Delta^*(f, g), h \rangle = \langle f \otimes g, \Delta h \rangle = \langle \Lambda_g(f), h \rangle = \langle f, \Gamma_g(h) \rangle = f(\Gamma_g(h)).$$

Then

$$\Delta h = \sum_i h^i \otimes \widehat{h}^i \implies \sum_i f(h^i) g(\widehat{h}^i) = f\left(\sum_i g(\widehat{h}^i) h^i\right),$$

so

$$\begin{aligned}\Delta h = \sum_i h^i \otimes \widehat{h}^i \implies \Gamma_g(h) &= \sum_i g(\widehat{h}^i) h^i = \sum_i (\mathrm{id} \otimes g)(h^i \otimes \widehat{h}^i) \\ &= (\mathrm{id} \otimes g)\left(\sum_i h^i \otimes \widehat{h}^i\right) = (\mathrm{id} \otimes g)\Delta h.\end{aligned}$$

Thus, we have shown that

$$\Gamma_g = (\mathrm{id} \otimes g) \circ \Delta.$$

In summary, we have a map $\Gamma : H^* \to \mathcal{L}(H)$.

Remark 1.2. The map $g \mapsto \Gamma_g$ does not act nicely with respect to the product structure on H^*.

7. Define $G_0 = \{g \in H^* : g(h_1 \cdot_m h_2) = g(h_1)g(h_2)$. We claim that G_0 is a group and $\Gamma_{g_1 \cdot_{\Delta^*} g_2} = \Gamma_{g_1} \circ \Gamma_{g_2}$ for $g_1, g_2 \in G_0$ (so $G = \{\Gamma_g L g \in G_0\}$ is a group). In the interest of time, we will not show this now.

Definition 1.3.

8. We say that our bialgebra $\mathcal{H}$ is **graded** if $\mathcal{H} = \oplus_{n \geq 0} \mathcal{H}_n$ with $m : \mathcal{H}_n \otimes \mathcal{H}_m \to \mathcal{H}_{n+m}$ and $\Delta : \mathcal{H}_n \to \oplus_{i+j=n} \mathcal{H}_i \otimes \mathcal{H}_j$ and **connected** if $\mathcal{H}_0 = \mathrm{span}\{\mathbf{1}\}$.

Theorem 1.1. *For a graded and connected bialgebra, an antipode S exists is unique, and $S : \mathcal{H}_n \to \mathcal{H}_n$. In fact,*

$$S = \sum_{k \geq 0} (\mathbf{1} \circ \mathbf{1}' - \mathrm{id})^{\cdot_m k}.$$